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A complete boundary integral formulation for compressible Navier–Stokes equations

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SUMMARY

A complete boundary integral formulation for compressible Navier–Stokes equations with time discretization by operator splitting is developed using the fundamental solutions of the Helmholtz operator equation with different order. The numerical results for wall pressure and wall skin friction of two-dimensional compressible laminar viscous flow around airfoils are in good agreement with field numerical methods. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: boundary integral formulation; boundary element method; compressible Navier–Stokes equations

INTRODUCTION

The boundary integral equation method is closely related to classical Green's function method. In the classical Green's function method, one applies the definition of the adjoint operator to a special function, which satisfies certain suitable boundary conditions to get an explicit expression for the solution. however, for non-linear problems as well as for linear problems with geometry of practical interest, obtaining an expression for the Green's function may be hard. It would be desirable to develop computational models of handling complexity, but based on cause-and-effect concepts accessible to the applications engineer. Such a project is offered by the new generation of boundary integral methods now starting to emerge. For non-linear problems, as in the case under consideration, the non-linear terms are formally treated as non-homogeneous terms. This yield the presence of domains integrals. In this paper the methods for transformation of domain integrals into boundary integrals presented in References [1, 2] are extended further and a complete boundary integral formulation for compressible Navier–Stokes equations with time discretization by operator splitting is developed. The advantages

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of complete boundary integral formulation are impressive: no mesh is needed external to the body boundary; very complex geometries can be treated; computation time are vastly smaller; conventional computers can be employed. The numerical results for the surface pressure and skin friction of airfoil given by present method show good agreement with field numerical methods.

THEORETICAL BASIS

The non-dimensional compressible Navier–Stokes equations are as follows:

$$\begin{aligned} &\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u}) = 0 \\ &\rho\partial\mathbf{u}/\partial t + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + (\gamma-1)T\nabla\rho = (1/Re)\{\nabla^2\mathbf{u} + (1/3)\nabla(\nabla\cdot\mathbf{u})\} \\ &\rho\partial T/\partial t + \rho\mathbf{u}\cdot\nabla T + (\gamma-1)\rho T\nabla\cdot\mathbf{u} = (1/Re)\{(\gamma/Pr)\nabla^2 T + F(\nabla u)\} \end{aligned} \tag{1}$$

where pressure p, density ρ, velocity $\mathbf{u}=\{u_i\}$, temperature T are non-dimensionalized by the free stream values $\rho_\infty|\mathbf{u}_\infty|^2$, ρ_∞, $\mathbf{u}_\infty$, and $|\mathbf{u}_\infty|^2/c_v$, respectively. Re, M_∞, Pr, c_v and γ are the Reynolds number, the free stream mach number, the Prandtl number, the specific heat at constant volume and the ratio of specific heat, respectively.

For two-dimensional flow:

$$F(\nabla\mathbf{u}) = (4/3)\{(\partial u_1/\partial x)^2 + (\partial u_2/\partial y)^2 - (\partial u_1/\partial x)(\partial u_2/\partial y)\} + (\partial u_1/\partial x + \partial u_2/\partial y)^2$$

where u_1 and u_2 are the velocity components along x and y directions. For simplicity, only Direchlet boundary conditions are considered.

On far field boundaries:

$$\begin{aligned} &\rho = 1 \\ &T = T\infty = 1/\gamma(\gamma-1)M_\infty^2 \\ &\mathbf{u} = \mathbf{u}_\infty \end{aligned} \tag{2}$$

On the rigid boundaries of body:

$$\begin{aligned} &|\mathbf{u}| = 0 \\ &T = T_B = T\infty\{1 + ((\gamma-1)/2)M_\infty^2\} \quad \text{(the free stream total temperature)} \end{aligned} \tag{3}$$

Since we consider time dependent equations, the initial conditions have also to be added:

$$\rho(\mathbf{r},0) = \rho_0(\mathbf{r}), \quad \mathbf{u}(\mathbf{r},0) = \mathbf{u}_0(\mathbf{r}), \quad T(\mathbf{r},0) = T_0(\mathbf{r})$$

In order to establish the complete boundary integral formulation for compressible Navier–Stokes equations, a new variable $\sigma = \ln\rho$ is introduced. With this variable, the compressible Navier–Stokes equations become,

$$\begin{aligned} &\partial\sigma/\partial t + \nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla\sigma = 0 \\ &\partial\mathbf{u}/\partial t - \mu\nabla^2\mathbf{u} + \beta\nabla\sigma = \psi(\sigma,\mathbf{u},T) \\ &\partial T/\partial t - \pi\nabla^2 T = \chi(\sigma,\mathbf{u},T) \end{aligned} \tag{4}$$

where $\mu = 1/Re$, $\pi = \gamma\mu/(RePr)$, $\beta = (\gamma-1)T_B = (1/\gamma)[(\gamma-1)/2 + 1/M_\infty^2]$

$$\psi(\sigma, \mathbf{u}, T) = -(\gamma-1)[\nabla T + (T - T_B)\nabla\sigma] - (\mathbf{u}\cdot\nabla)\mathbf{u} + (1/Re)\{e^{-\sigma}(\nabla^2\mathbf{u} + 1/3\nabla(\nabla\cdot\mathbf{u})) - \nabla^2\mathbf{u}\}$$

$$\chi(\sigma, \mathbf{u}, T,) = -(\gamma-1)T\nabla\cdot\mathbf{u} - \mathbf{u}\cdot\nabla T + \gamma/(RePr)(e^{-\sigma}-1)\nabla^2 T + (1/Re)e^{-\sigma}F(\nabla\mathbf{u})$$

Using time discretization by operator splitting methods, we should obtain the following θ scheme [3] from Equation (4). In this paper we take $\theta = \frac{1}{4}$. For $n \geqslant 0$, starting from $\sigma^n, \mathbf{u}^n, T^n$ we solve

$$(\sigma^{n+1/4} - \sigma^n)/(\Delta t/4) + \nabla\cdot\mathbf{u}^{n+1/4} = -\mathbf{u}^n\cdot\nabla\sigma^n \tag{5a}$$

$$(\mathbf{u}^{n+1/4} - \mathbf{u}^n)/(\Delta t/4) - a\mu\nabla^2\mathbf{u}^{n+1/4} + \beta\nabla\sigma^{n+1/4} = \psi(\sigma^n, \mathbf{u}^n, T^n) + b\mu\nabla^2\mathbf{u}^n \tag{5b}$$

$$(T^{n+1/4} - T^n)/(\Delta t/4) - a\pi\nabla^2 T^{n+1/4} = \chi(\sigma^n, \mathbf{u}^n, T^n) \tag{5c}$$

$$(\sigma^{n+3/4} - \sigma^{n+1/4})/(\Delta t/2) + \mathbf{u}^{n+3/4}\nabla\sigma^{n+3/4} = -\nabla\cdot\mathbf{u}^{n+1/4} \tag{6a}$$

$$(\mathbf{u}^{n+3/4} - \mathbf{u}^{n+1/4})/(\Delta t/2) - b\mu\nabla^2\mathbf{u}^{n+3/4} - \psi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4}) = a\mu\nabla^2\mathbf{u}^{n+1/4} - \beta\nabla\sigma^{n+1/4} \tag{6b}$$

$$(T^{n+3/4} - T^{n+1/4})/(\Delta t/2) - b\pi\nabla^2 T^{n+3/4} - \chi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4}) = a\pi\nabla^2 T^{n+1/4} \tag{6c}$$

$$(\sigma^{n+1} - \sigma^{n+3/4})/(\Delta t/4) + \nabla\cdot\mathbf{u}^{n+1} = -\mathbf{u}^{n+3/4}\cdot\nabla\sigma^{n+3/4} \tag{7a}$$

$$(\mathbf{u}^{n+1} - \mathbf{u}^{n+3/4})/(\Delta t/4) - a\mu\nabla^2\mathbf{u}^{n+1} + \beta\nabla\sigma^{n+1} = \psi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4}) + b\mu\nabla^2\mathbf{u}^{n+3/4} \tag{7b}$$

$$(T^{n+1} - T^{n+3/4})/(\Delta t/4) - a\pi\nabla^2 T^{n+1} = \chi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4}) + b\pi\nabla^2 T^{n+3/4} \tag{7c}$$

with $0 < a$, $b < 1$, $a + b = 1$ for $\theta = \frac{1}{4}$:

$$a = (1-2\theta)/(1-\theta) = 2/3, \quad b = \theta/(1-\theta) = 1/3 \tag{8}$$

It can be seen that at both time step $n + 1/4$ and $n + 1$ all require the solution of two same systems of couple Equations (5a), (5b) and (7a), (7b). They can be written as:

$$\lambda\sigma + \nabla\cdot\mathbf{u} = g \tag{9}$$

$$\lambda\mathbf{u} - 2/3\mu\nabla^2\mathbf{u} + \beta\nabla\sigma = f \tag{10}$$

where $\lambda = 1/(\Delta t/4)$, g and f are known functions of σ, $\mathbf{u}$ and T at previous time step.

$$g = -\mathbf{u}^n \cdot \nabla\sigma^n + \lambda\sigma^n \quad \text{(for Equation (5a))}$$
$$= -\mathbf{u}^{n+3/4} \cdot \nabla\sigma^{n+3/4} + \lambda\sigma^{n+3/4} \quad \text{(for Equation (7a))} \tag{11}$$
$$f = \psi(\sigma^n, \mathbf{u}^n, T^n) + (1/3)\mu\nabla^2\mathbf{u}^n + \lambda\mathbf{u}^n \quad \text{(for Equation (5b))}$$
$$= \psi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4}) + 1/3\mu\nabla^2\mathbf{u}^{n+3/4} + \lambda\mathbf{u}^{n+3/4} \quad \text{(for Equation (7b))} \tag{12}$$

Taking the divergence of both sides in Equation (10), we have

$$\lambda\nabla \cdot \mathbf{u} - 2/3\mu\nabla^2(\nabla \cdot \mathbf{u}) + \beta\nabla^2\sigma = \nabla \cdot f \tag{13}$$

On the other hand, Equation (9) yields

$$\nabla \cdot \mathbf{u} = g - \lambda\sigma \tag{14}$$

Combining Equations (13) and (14), we obtain

$$\lambda_1\sigma - \nabla^2\sigma = f_1 \tag{15}$$

with $\lambda_1 = \lambda^2/(\beta + (2/3)\lambda\mu)$, $f_1 = (\lambda g - \nabla \cdot f - (2/3)\lambda\mu\nabla^2 g)/(\beta + (2/3)\lambda\mu)$ in order to have a well posed problem in σ, it is necessary to have an additional boundary condition of type: $\sigma = k$ on body. After computing σ from Equation (15), $\mathbf{u}$ may be solved from Equation (10) which is now reduced to the same type as Equation (15) with the boundary condition (2)–(3) and then the value of k has to be calculated in order that Equation (9) is satisfied. Equations (5c) and (7c) already take the type as Equation (15). A linear variant of Equations (6b) and (6c) are obtained by substituting $\psi(\sigma^{n+1/4}, \mathbf{u}^{n+1/4}, T^{n+1/4})$ for $\psi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4})$ in Equation (6b) and $\chi(\sigma^{n+1/4}, \mathbf{u}^{n+1/4}, T^{n+1/4})$ for $\chi(\sigma^{n+3/4}, \mathbf{u}^{n+3/4}, T^{n+3/4})$ in Equation (6c). After these substitutions, Equations (6b) and (6c) are also reduced to the type as Equation (15). Hence, the problem for the solution of compressible Navier–Stokes equations are now really reduced to the problems for the solution of a series of equation with the type of Equation (15). Equation (15) can be solved by following fundamental solution method. Multiplying Equation (15) with the fundamental solution H_0 of Helmholtz operator equation with order zero and integrate it with respect to domain Ω, we have

$$\int_\Omega (\lambda_1\sigma - \nabla^2\sigma)H_0 \, d\Omega = \int_\Omega f_1 H_0 \, d\Omega \tag{16}$$

where H_0 satisfies equation:

$$(\lambda_1 - \nabla^2)H_0 = \delta(\mathbf{r}) \tag{17}$$

Here δ is the impulse function, $\mathbf{r}$ is the position vector.
According to the Green theorem,

$$\int_\Omega H_0\nabla^2\sigma \, d\Omega = \int_B (H_0\partial\sigma/\partial n - \sigma\partial H_0/\partial n)\, dB + \int_\Omega \sigma\nabla^2 H_0 \, d\Omega \tag{18}$$

Substituting Equation (18) into Equation (16), we have

$$\int_\Omega \sigma(\lambda_1 - \nabla^2)H_0\,\mathrm{d}\Omega = \int_B (H_0\partial\sigma/\partial n - \sigma\partial H_0/\partial n)\,\mathrm{d}B + \int_\Omega f_1 H_0\,\mathrm{d}\Omega \tag{19}$$

where B are the boundaries of domain Ω. Substituting Equation (17) into the left-hand side of Equation (19) and considering the integrating properties of impulse function δ, we have

$$c\sigma(\mathbf{r}) = \int_B (H_0\partial\sigma/\partial n - \sigma\partial H_0/\partial n)\,\mathrm{d}B + \int_\Omega f_1 H_0\,\mathrm{d}\Omega \tag{20}$$

c is a coefficient, for smooth boundary $c=\frac{1}{2}$. In order to transform the domain integral $\int_\Omega f_1 H_0\,\mathrm{d}\Omega$ in Equation (20) into a series of boundary integrals, two new functions A_0 and H_1 are first introduced. $A_0 = f_1$, $H_0 = (\lambda_1 - \nabla^2)H_1$. Thus,

$$\int_\Omega f_1 H_0\,\mathrm{d}\Omega = \int_\Omega A_0(\lambda_1 - \nabla^2)H_1\,\mathrm{d}\Omega \tag{21}$$

According to the Green theorem

$$\int_\Omega A_0\nabla^2 H_1\,\mathrm{d}\Omega = \int_B (A_0\partial H_1/\partial n - H_1\partial A_0/\partial n)\,\mathrm{d}B + \int_\Omega H_1\nabla^2 A_0\,\mathrm{d}\Omega$$

Hence,

$$\int_\Omega f_1 H_0\,\mathrm{d}\Omega = \int_\Omega A_0(\lambda_1 - \nabla^2)H_1\,\mathrm{d}\Omega = \int_\Omega H_1(\lambda_1 - \nabla^2)A_0\,\mathrm{d}\Omega - \int_B (A_0\partial H_1/\partial n - H_1\partial A_0/\partial n)\,\mathrm{d}B \tag{22}$$

Similarly, if we set $A_1 = (\lambda_1 - \nabla^2)A_0$, $H_1 = (\lambda_1 - \nabla^2)H_2$, then the domain integral on the right-hand side of Equation (22) can also be rewritten as

$$\int_\Omega H_1(\lambda_1 - \nabla^2)A_0\,\mathrm{d}\Omega = \int_B (A_1\partial H_2/\partial n - H_2\partial A_1/\partial n)\,\mathrm{d}B + \int_\Omega H_2(\lambda_1 - \nabla^2)A_1\,\mathrm{d}\Omega \tag{23}$$

The procedure can be generalized by introducing two sequence of functions defined by the following recurrence formulae

$$A_{j+1} = (\lambda_1 - \nabla^2)A_j, \quad H_j = (\lambda_1 - \nabla^2)H_{j+1}, \quad j = 0, 1, 2, \ldots \tag{24}$$

Thus the domain integral $\int_\Omega f_1 H_0\,\mathrm{d}\Omega$ in Equation (20) can be expressed as the summations of infinite boundary integrals

$$\int_\Omega f_1 H_0\,\mathrm{d}\Omega = \sum_{j=0}^{\infty} \int_B (A_j\partial H_{j+1}/\partial n - H_{j+1}\partial A_j/\partial n)\,\mathrm{d}B \tag{25}$$

More generally, the jth order fundamental solution of Helmholtz equation Hj satisfies

$$(\lambda_1 - \nabla^2)H_j = H_{j-1}, \quad j = 1, 2, \ldots$$

and can be expressed as [4]

$$\begin{aligned} B_0 &= 1/(2\pi) \\ H_0 &= B_0 K_0(\lambda_1^{1/2} r) \\ B_j &= B_{j-1}/(2j\lambda_1) = B_0/((2\lambda_1)^j j!) \\ H_j &= B_j(\lambda_1^{1/2} r)^j K_j(\lambda_1^{1/2} r) = B_0 r^j K_j(\lambda_1^{1/2} r)/((2\lambda_1^{1/2})^j j!), \quad j = 1, 2, \ldots \end{aligned} \tag{26}$$

where $K_j(x)$ represents the second kind modified Bessel function of jth order. Substituting Equation (25) into Equation (20), a complete boundary integral formulation for Equation (15) is finally obtained and can be solved by well known boundary element method.

$$c\sigma(r) = \int_B (H_0 \partial\sigma/\partial n - \sigma \partial H_0/\partial n)\,\mathrm{d}B + \sum_{j=0}^{\infty} \int_B (A_j \partial H_{j+1}/\partial n - H_{j+1} \partial A_j/\partial n)\,\mathrm{d}B \tag{27}$$

Notice that the introduction of factor $(2\lambda_1^{1/2})^j j!$ into the denominator of expression H_j guarantees the rapid convergence of Equation (27) as j increase, especially for small Δt and the flow with higher Reynold's number because the smaller the Δt and the higher the Reynold's number, the larger the λ_1 will be.

NUMERICAL RESULTS AND CONCLUDING REMARKS

In 1987, a GAMM-workshop was organized to bring a small number of scientists highly concerned with the numerical solution of the compressible Navier–Stokes equations to calculate the assigned test problems [5] and to compare the results presented by the contributors each other. One of the assigned test problem was external 2D flow around a NACA0012 airfoil with Direchlet body boundary condition at $M_\infty = 0.8$, $Re = 73$ and 500, respectively, angles of attack $\alpha = 10°$. All the methods used by the contributors in Reference [5] were field method (finite differences, finite elements and finite volumes). In order to compare the results given by present complete boundary integral method with the results [6] given in Reference [5] the same test problems are calculated in this paper. One of the contributors [6] in Reference [5] solved the problems by using a new explicit Navier–Stokes code based on a combination of central finite differencing and rational Rung–Kutta time stepping. It is a more accurate field method. So it's results is used for comparison. Figures 1 and 2 show the laminar viscous wall pressure coefficient and skin friction coefficient on NACA 0012 airfoil calculated by present method and the results of field method given in Reference [6]. No field values (such as streamlines around airfoil, etc.) are compared because the results of the solution of boundary integral formulation are the values of variables on the wall boundary.

For the flow with $M_\infty = 0.8$ and $Re = 73$, if we take time step $\Delta t = 0.1$ then we have $\mu = 0.014$, $\beta = 1.26$, $\lambda = 40$, $\lambda_1 = 979.792$. The relationships between j and $H_j/(r^j K_j)$ are as follows:

j	$H_j(r^j K_j)$
0	0.159

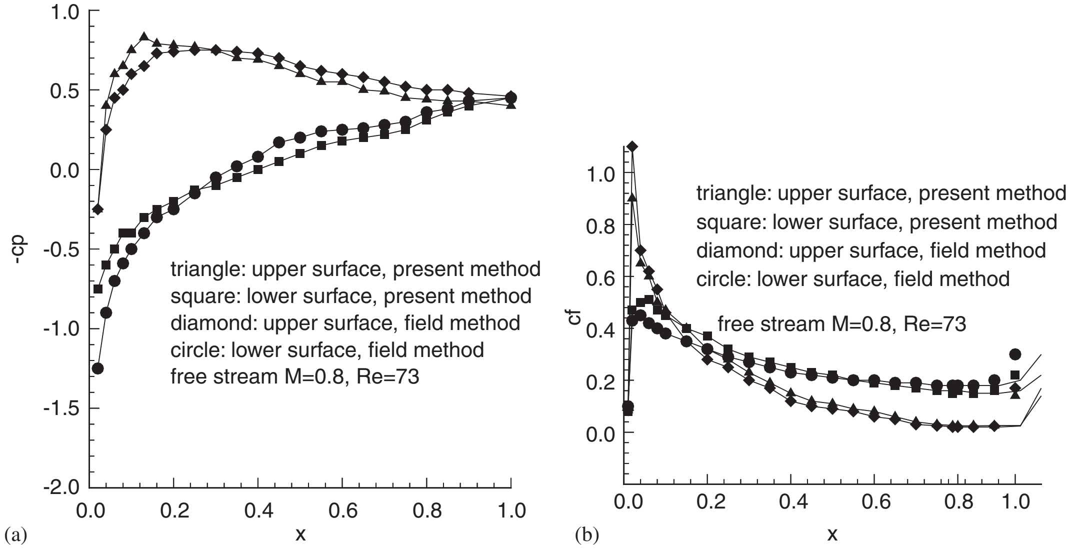

Figure 1. (a) Surface pressure coefficient cp; and (b) skin friction coefficient cf.

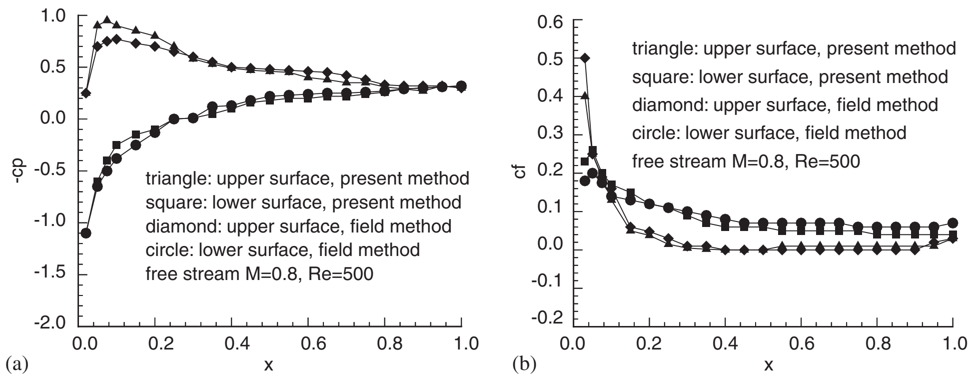

Figure 2. (a) Surface pressure coefficient cp; and (b) skin friction coefficient cf.

1	2.540×10^{-3}
2	2.029×10^{-5}
3	1.080×10^{-7}
4	4.313×10^{-10}
5	1.378×10^{-12}

The solutions are convergent at $j = 4$ and the maximum value of the relative difference of pressure coefficient between j and $j-1$ $(cp_j - cp_{j-1})/cp_j$ is less than 10^{-7}.

For the flow with $M_\infty = 0.8$ and $Re = 500$, if we take time step $\Delta t = 0.1$ then we have $\mu = 0.002$, $\beta = 1.26$, $\lambda = 40$, $\lambda_1 = 1218.58$. The relationships between j and $H_j/(r^j K_j)$ are as follows:

j	$H_j/(r^j K_j)$
0	0.159
1	2.277×10^{-3}
2	1.631×10^{-5}
3	7.787×10^{-8}
4	2.788×10^{-10}
5	7.988×10^{-13}

The solutions are convergent at $j = 3$ and the maximum value of the relative difference of pressure coefficient between j and $j - 1$ $(cp_j - cp_{j-1})/cp_j$ is less than 10^{-7}. The computing results show good agreement with the field method [6]. It can also be seen that even for low Reynold's number, the solution can still be converged at a small number of j. Obviously, the number of j for convergence will be reduced as the time step is further reduced.

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